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STABILITY IN THE CASE OF A NEUTRAL LINEAR APPROXIMATION

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As is known, linearization is the basic method of studying the stability of singular points of a system of ordinary differential equations. The linearized systems can be decomposed, in general, into three kinds of components: stable, neutral, unstable. If an unstable component is present the question of stability is decided negatively regardless of the other components. In the presence of only one stable component the original system is found to be stable. The proof of this assertion is the concern of Lyapunov's stability theory. If besides a stable one there is also a neutral component, the question remains open.

Without loss of generality we can consider that only a neutral component is present, since the general case can be reduced to this more simple case in the following simple way. We assume that in the original system all the variables belonging to stable components are zero. There results a "reduced" system having only neutral components. It can be shown that if the "reduced" system turns out to be stable, then the original system is also stable. The basic difficulty in the study of a system with a neutral component is that the motion near the singular point is almost periodic in the principal term, while stability or instability becomes apparent only in subsequent approximations.

Therefore to settle the question of stability it is necessary to carry out the subdivision of motions [1] and to eliminate the basic motion. Let us investigate a system having the form

$$\frac{du}{dt} = U(u), \tag{1}$$

and assume that $u = u_0 \equiv$ constant is the solution of system (1). Translating the origin of coordinates to the point u_0, we consider $u = u_0 + \epsilon x$ for small values of the parameter ϵ which corresponds to small neighborhoods of u_0 in the original variables. System (1) in the new variables has the form

$$\frac{du}{dt} = A(x) + \epsilon A_1(x) + \frac{\epsilon^2}{2!} A_2(x) + \cdots. \tag{2}$$

Here $A, A_1, A_2, \cdots$ are homogeneous polynomials respectively of the first, second, third, etc. degrees. According to the basic assumption the matrix A has pure imaginary eigenvalues. We apply the scheme of subdivision of motions to system (2). This implies that we must introduce the change of variables

$$y = x + \epsilon Q_1(x) + \frac{\epsilon^2}{2!} Q_2(x) + \cdots \tag{3}$$

so that the new system

$$\frac{dy}{dt} = A(y) + \epsilon B_1(y) + \frac{\epsilon^2}{2!} B_2(y) + \cdots$$

permits subdivision of motions. In [1] it is shown that the functions Q_n and B_n can be computed from the formulas

$$B_n(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\frac{\partial f}{\partial y}\right)^{-1} C_n(f)\, dt,$$

$$Q_n(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T (T-t)\left(\frac{\partial f}{\partial y}\right)^{-1} [C_n(f) - B_n(f)]\, dt. \tag{6}$$

In these formulas $f = f(y, t)$ is the solution of the unperturbed equation $\partial f/\partial t = A(f)$ satisfying the initial condition $f(y, 0) = y$. The function C_n can be computed if we already know B_{n-1}, Q_{n-1}. For example, $C_1(x) = A_1(x)$, $C_2(x) = A_2(x) + 2[(dQ_1/dx)\, A_1 - (dB_1/dx)\, Q_1] - (d^2 A/dx^2)\, Q_1 Q_1$, etc. In our case it is not difficult to show that when A_n is a homogeneous polynomial of degree $(n+1)$, the functions C_n, B_n, Q_n are also homogeneous polynomials of degree $(n+1)$ in their own respective arguments.

The function $B_n(y)$ is of fundamental interest to us since it determines the stability of the system. The simplest form of this function is obtained when the matrix A is diagonal. Since matrix A by hypothesis has pure imaginary eigenvalues, both the variables and the coefficients of the equations are necessarily complex numbers. The coefficients are not, of course, arbitrary complex numbers, because the system is obtained from a real system. If we denote by the variable x^{α^*} the complex conjugate of x^α, it is easy to see that the coefficients satisfy conditions of the type $A^{\alpha^*}_{\beta^*\gamma^*} = \overline{A^{\alpha}_{\beta\gamma}}$. Analogous equalities hold for the coefficients of the polynomials B, C and Q.

Let us denote by $i\omega_\alpha$ the eigenvalues of matrix A and observe that

$$\omega_{\alpha^*} = -\,\omega_\alpha. \tag{7}$$

For a diagonal matrix the computations in formula (5) are not difficult to carry out to the end. In fact, in this case the solution $f(y, t)$ has the form $f^\alpha = y^\alpha \exp(i\omega_\alpha t)$. Substituting in (5) we find $B_{\beta\gamma} y^\beta y^\gamma = C^\alpha_{\beta\gamma} y^\beta y^\gamma \lim_{T\to\infty} \int_0^T \exp(it(\omega_\beta + \omega_\gamma - \omega_\alpha))\, dt$. But the resulting limit is different from zero (and equal to unity) only if the inequality

$$\omega_\alpha - (\omega_\beta + \omega_\gamma) = 0, \tag{8}$$

is satisfied, which it is natural to call the *condition of internal resonance of the second order.* In exact analogy, when we compute B_2 only those terms will remain for which we have *resonance of the third order*

$$\omega_\alpha - (\omega_\beta + \omega_\gamma + \omega_\delta) = 0. \tag{9}$$

We note that in any system there is necessarily resonance of the third order. In fact, setting $\beta = \alpha$, $\delta = \gamma^*$ we see that (9) follows from (7). Resonance of the second order is another matter: as a rule it is absent. This important conclusion can be formulated more precisely in the following way.

Let us consider a system containing a parameter. In this case the fundamental frequency of the system is ω_α and hence the combinative frequencies $\omega_{\alpha\beta\gamma}$, $\omega_{\alpha\beta\gamma\delta}$ (the right hand sides of (8) and (9)) are functions of this parameter. Certain of these combinative frequencies, namely $\omega_{\alpha\alpha\gamma\gamma^*}$, equal zero for all values of the parameter. This is trivial resonance of the third order. Other frequencies of the second order, in particular $\omega_{\alpha\beta\gamma}$, reduce to zero only at isolated points, determining the critical values of the parameter. The very interesting question of the passage of the system through such critical states (in particular for a zero root of matrix A, when (8) follows from (7)) is not studied here. The present note deals with the analysis of the general case when the system has no resonances except the trivial one. Then, the case reduces to the study of a system of equations of the form (where summation is to be carried out only over the index α):

$$\frac{d\eta^k}{dt} = -\eta^k(E_{k\alpha}\eta^\alpha), \tag{10}$$

where $E_{\alpha\beta} = -(\epsilon^2/2!)(B^\alpha_{\alpha\beta\beta^*} + B^{\alpha^*}_{\alpha^*\beta\beta^*})$. This system can be obtained from (4) if instead of y^k we introduce a new variable η^k by the formula $\eta^k = |y^k|^2 = y^k y^{k*}$, and have half as many equations as (4). *The question of the stability of system* (4) *thus reduces to the question of the stability of system* (10) *in the cone* $\eta^k \geq 0$.

Note in this connection one essential property of system (10). If somehow the variable η^k equals zero for $t = t_0$ then it is equal to zero identically; this follows easily from the form of system (10). Therefore, in particular, any variable maintains its sign for all values of t. If we assume that all the variables except one are equal to zero, then it is not difficult to obtain the necessary stability condition

$$E_{kk} > 0. \tag{11}$$

The positive-definite, symmetric part of matrix $E_{\alpha\beta}$

$$\left(\left(\frac{E_{\alpha\beta} + E_{\beta\alpha}}{2}\right)\right) > 0 \tag{12}$$

is the simplest sufficient condition for stability. This assertion is easily shown by adding together all the solutions of system (10).

Let us pass to the establishment of the necessary and sufficient conditions. As in linear systems, the invariant paths of the system, i.e., the solutions of the form $\eta^k = \eta_0^k \eta(t)$, are important in the formulation of the stability criterion. Substituting these expressions into (10) we have

$$\frac{d\eta}{dt} = -E\eta^2, \quad \eta(0) = 1, \tag{13}$$

$$\eta_0^k[E_{k\alpha}\eta_0^\alpha - E] = 0. \tag{14}$$

Here E is a parameter analogous to an eigenvalue in linear systems. However, the magnitude of this parameter, in view of its proportionality to the length of the given initial vector η_0^α, is of no importance whatsoever. As is evident from equation (13), it is the sign of E that is important since it determines stability.

The method of finding all the invariant paths of system (10) is suggested by the form of system (14). At first we retain in every one of the equations only the second factor. There results the basic system of linear equations

$$E_{k\alpha}\eta_0^\alpha = E. \tag{15}$$

If the matrix $E_{\alpha\beta}$ is not degenerate, then for any value of the parameter E system (15) has a unique solution. These solutions fill out the invariant curve consisting of two paths, the stable one ($E > 0$) and the unstable one ($E < 0$). However, if the matrix is degenerate, the solution exists only for $E = 0$, and then it is determined to within a proportional factor. Again we have an invariant path, this time neutral. All solutions of the nonlinear system (14) can be obtained by retaining in every equation either the first or the second factor. Therefore, altogether 2^n solutions are obtained, including those considered above and the trivial one: $\eta^k = 0$. It is easy to see that this procedure corresponds sponds to an independent study of system (10) on all possible surfaces of the cone $\eta^k \geq 0$. An understanding of the invariant paths permits the formulation of

The stability criterion. For system (10) to be stable in the cone $\eta^k \geq 0$ it is necessary and sufficient that there not be a single neutral or unstable path inside and on the boundary of the cone.

The necessity is obvious.

As in linear systems, sufficiency can be proved by constructing a Lyapunov function. We indicate a guide to the idea of this construction. Rewrite system (10) in the form $d \ln \eta^k / dt = - E_{k\alpha} \eta^\alpha$. Now multiply both sides of every one of the equations by z_k and add all the n equations. We obtain the relation

$$\frac{d \ln \Phi}{dt} = - \Psi. \tag{16}$$

Here we introduce the notation

$$\ln \Phi = \sum_k z_k \ln \eta^k, \qquad \Psi = \sum_\alpha \zeta_\alpha \eta^\alpha, \qquad \zeta_\alpha = \sum_k E_{k\alpha} z^k.$$

For the fulfillment of the conditions of the criterion we can show that there exists a positive z^k, to which corresponds a positive ζ_α. The resulting function Φ cannot be used as a Lyapunov function only because it vanishes on the boundary of the positive cone. This defect can be corrected by adding to Φ a bounded Lyapunov function (of sufficiently small weight so as not to spoil the negative definiteness of its derivative). We assume that the existence of a Lyapunov function of smaller dimension has been proved (by induction).

In conclusion we analyze [2] a system of four equations (two degrees of freedom in the theory of oscillations). In this case equation (10) can be integrated by quadrature. By changing the scale it is possible to make $E_{11} = 1$, $E_{22} = 1$. Therefore this system generates a two-parameter family which can be mapped by points on the plane (α, β) where $\alpha = E_{12}$, $\beta = E_{21}$.

It is not difficult to show that unstable systems lie below the negative branch of the hyperbola $\alpha\beta - 1 = 0$. Above the line $\alpha + \beta + 2 = 0$ lies the region of monotonic stability. Formally, between these curves lie the stable systems the solutions of which can, however, increase before they begin to decay. The degree of "upswing" in this system (i.e., the possible initial increase in the oscillation amplitude) can be roughly estimated from the number $|\alpha| + |\beta|$. It is clear that if the "upswing" is large the system is unstable for all practical purposes.

For a system with two frequencies it is not difficult to construct immediately a linear Lyapunov function in the variables η_1, η_2. Therefore, together with one-frequency systems, we can use as a starting point the proof by induction for general systems.

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